Eigenmode Analysis in Unsteady Aerodynamics: Reduced-Order Models

Earl H. Dowell*
Duke University, Durham, North Carolina 27708-0271

A conceptually novel and computationally efficient technique for computing unsteady flow about isolated airfoils, wings, and turbomachinery cascades is presented. Starting with either a time-domain or frequency-domain computational fluid dynamics analysis of unsteady aerodynamic or aeroacoustic flows, a large, sparse eigenvalue problem is solved using the Lanczos algorithm. Then, using just a few of the resulting eigenmodes, a reduced-order model of the unsteady flow is constructed. With this model, one can rapidly and accurately predict the unsteady aerodynamic response of the system over a wide range of reduced frequencies. Moreover, the eigenmode information provides important insights into the physics of unsteady flows. Finally, the method is particularly well suited for use in the active control of aeroelastic and aeroacoustic phenomena, as well as in standard aeroelastic analysis for flutter or gust response. Numerical results to be presented include 1) comparison of the reduced-order model to classical unsteady incompressible aerodynamic theory, 2) reduced-order calculations of compressible unsteady aerodynamics based on the full potential equation, 3) reduced-order calculations of unsteady flow about an isolated airfoil based on the Euler equations, and 4) flutter analysis using the reduced-order model.

Introduction

WHY study the eigenmodes of unsteady aerodynamic flows? This is perhaps the fundamental question most often asked, although occasionally someone will express surprise that eigenmodes even exist for these flows. There are several reasons:

1) Eigenvalues and eigenmodes for these flows do exist! Perhaps they can tell us something about the basic physical behavior of the flowfield.

2) Indeed, if a relative small number of eigenmodes are dominant, this immediately suggests a way to construct an efficient computational aerodynamic model using these dominant modes.

3) Finally, by constructing the aerodynamic model in eigenmodal form, it is particularly user friendly in combining the eigenmode aerodynamic model with structural modal models to form an aeroelastic modal model with a minimum number of degrees of freedom for a given desired level of accuracy. These aeroelastic models will be especially attractive for design studies including the active control of such systems.

This paper is intended to provide an overview and perspective for the future. It is based largely on the work described in Refs. 1–6; earlier work is noted in those references.

Brief and Selected Account of Landmarks in Unsteady Aerodynamics

Here several landmarks in unsteady aerodynamics are touched upon. They anticipate (at least in retrospect) some aspects of the eigenmode approach to unsteady aerodynamics that will be presented in this paper. The account reflects that prior work that has been most helpful to the author and his colleagues in developing the eigenmodal approach.

Hydrodynamic Stability Theory

One of the classical eigenmode analyses in time-dependent fluid mechanics is that which considers the stability of laminar, viscous flows. Dating from the work of Rayleigh, Heisenberg, and Lin this subject has a rich heritage. See Lin[7] for a clear and elegant discussion of the early literature. Perhaps because of the perceived subtlety

and difficulty of the hydrodynamic stability theory, the eigenmode approach has rarely been pursued for convecting aerodynamic flows.

Classical Acoustic and Hydrodynamic Eigenmodes for Contained Fluids

This literature is quite extensive though it has not provided an impetus for extension to convecting, unbounded flows. There is a small, but interesting body of work on nonlinear dynamical models based upon eigenmodes. See, for example, the monograph edited by Abramson.[8]

Classical Linear Airfoil Theory

Glauret's theory and Theodorsen's extension to unsteady aerodynamic flows have echoes of an eigenmode approach, at least as viewed from the present vantage point. Two aspects of the classical theory deserve to be highlighted.

The classical airfoil theory is indeterminable, without the addition of an empirical constraint condition, the Kutta condition. That is, there is a nontrivial pressure distribution (an eigenmode!) that gives rise to a zero downwash or airfoil motion. To find the strength of this special pressure distribution, a special condition, the Kutta condition, must be imposed to make the solution unique.

Another point to recall is that the Theodorsen function has a branch cut in the complex frequency domain. This branch cut will appear in a certain guise in the eigenvalue distributions to be discussed in a later section of this paper. See the discussion in any one of several standard texts.[9-13]

Singular Integral Equation Formulation for Classical, Compressible, Potential Flow over Wings

It is well known that this theory has the same basic singularities in the kernel of the integral equation as the classical airfoil theory. This suggests its eigenspectrum will be similar to that of the airfoil, though there is an interesting surprise in the eigenspectrum of wings as compared to that of the airfoil.[9-13]

Computational Fluid Dynamics Models

Perhaps not surprisingly, it is in someways easiest to see the value of an eigenvalue/eigenmode approach from examining computational fluid dynamics (CFD) models. This is because a typical CFD model is derived from a finite difference (element, volume) approximation to a set of partial differential equations (PDEs) in time and space that reduces the set of PDEs to a much larger set of ordinary differential equations (ODEs) in time.

Once one has a set of ODEs, it is not a great leap to think of eigenvalues and eigenvectors. Although, ironically, it may be easier to make that leap from a mathematical or dynamical system point of view than from a consideration of classical aerodynamic theory per se. For a representative discussion of the CFD literature in the context of unsteady aerodynamics, see Refs. 14 and 15.

Perhaps it is not surprising that the use of eigenmodes in unsteady aerodynamics has come about at this time in history. Although similar work in the context of classical aerodynamic theory could have been done some years ago.

Eigenmodes and Reduced-Order Models

The details vary from one level of fluid model to another as to how one determines the eigenvalues and eigenmodes and then constructs the reduced order model. Rather than repeat the discussion for each specific fluid model, here the discussion is presented in generic form. It is the one that most closely follows the calculation for the Euler or Navier–Stokes equations. For specific calculations for a vortex lattice model[1] or a full potential model[2-4] the reader is referred to the literature. The following discussion is taken from Ref. 5.

The Euler and Navier–Stokes equations, represent a system of highly nonlinear partial differential equations, which can be written at each point in the interior of a computational flow field domain as $\bar{q}_{i_\tau} = Q(\bar{q}_i, \bar{q}_e) = Q(\bar{q})$ or in discrete time as

$$\frac{\Delta \bar{z}_i^{n+1}}{\Delta \tau} = Q(\bar{q}_i^n, \bar{q}_e^n) \tag{1}$$

where $\bar{q}_i$ represents flow variables at the interior of the computational domain, and $\bar{q}_e$ represents those on the exterior (boundary) of the computational domain.

There are N interior flow equations, with four equations [for a 2D Euler flow] written at each of the $\frac{N}{4}$ interior grid points. The four unknowns at each grid point are [pressure, two components of momentum and energy for a 2D Euler flow]. Additionally, there are M nonlinear algebraic boundary condition equations, also with four equations written at each of the $\frac{M}{4}$ grid points on the boundary of the computational domain.

$$P(\bar{q}_i^n, \bar{q}_e^n, \alpha^n) = 0 \tag{2}$$

where there are L α-variables, related to airfoil shape, or motion, etc. Eqs. 1 and 2 form a system of $N + M$ equations and $N + M$ unknowns.

Now, assuming that the unsteady flow field is a result of small dynamic perturbations about steady state, $(\bar{q} = \bar{q}_0 + \hat{q}(\tau), \alpha = \alpha_0 + \hat{\alpha}(\tau))$ and noting that $Q(\bar{q}_0) = P(\bar{q}_0, \alpha_0) = 0$. Equations 1 and 2 become

$$\frac{\Delta \hat{q}_i^{n+1}}{\Delta \tau} = A \hat{q}_i^n + B_i \hat{q}_e^n \tag{3}$$

$$C \hat{q}_e^n - D \hat{q}_i^n - E \hat{\alpha}^n = 0. \tag{4}$$

The exterior degrees of freedom can be eliminated from Eq. 3 by first solving Eq. 4 for $\hat{q}_e$.

$$\hat{q}_e^n = C^{-1} D \hat{q}_i^n + C^{-1} E \hat{\alpha}^n \tag{5}$$

The following system of N coupled equations, which govern the linearized system response, is obtained by noting that the perturbation flow field at time step $n+1$ is $\hat{q}^{n+1} = \hat{q}^n + \Delta \hat{q}^{n+1}$, and then utilizing Eqs. 3 and 5 to give

$$\hat{q}^{n+1} = A_{eq} \hat{q}_1^n + B_{eq} \hat{\alpha}^n \tag{6}$$

where

$$A_{eq} \equiv I + \Delta\tau(A + BC^{-1}D) \tag{7}$$

$$B_{eq} \equiv \Delta\tau BC^{-1}E. \tag{8}$$

Note that the second term on the right hand side of Eq. 6 represents a known forcing term [for prescribed airfoil or wing motion]. Also, in Eq. 7, A_{eq} is shown to be a combination of conditions at the interior degrees of freedom, A, plus a contribution from the boundary, while in Eq. 8, B_{eq}, and therefore the forcing, depends only on conditions associated with the exterior

degrees of freedom. Finally, flow values on the airfoil boundary (for the calculation of lift, for example) can be obtained directly by Eq. 5.

Now, for zero forcing $(\hat{\alpha}^n \equiv 0)$, assume that the flow field at time n is related to an initial flow field in the manner that $\hat{q}_i^n = z^n \bar{q}$. Then, Eq. 6 reduces to

$$A_{eq} \bar{q} = z\bar{q}, \tag{9}$$

which represents an eigenvalue problem. Since A_{eq} is nonsymmetric, it will have complex eigenvalues, $z_1, z_2, z_3, \ldots, z_N$, as well as sets of both right eigenvectors, $[e^R]$ and left eigenvectors, $[e^L]$, which are biorthonormal. This dynamic system will be stable if the largest eigenvalue has magnitude less than 1.

The reduced coordinate $\hat{s}$ is defined such that

$$\hat{q}_i \equiv [e^R]\hat{s}. \tag{10}$$

Now, the governing system of equations can be decoupled by substituting Eq. 10 into Eq. 6, premultiplying by $[e^L]^T$, and taking advantage of the fact that $[e^L]^T[e^R] = [I]$ and $[e^L]^T[A_{eq}][e^R] = [Z]$. Additionally, by utilizing only the R largest magnitude eigenmodes in this transformation, a system of R decoupled equations $(R \ll N)$, the reduced order approximation of the governing equations, is obtained.

$$\hat{s}^{n+1} = Z_R \hat{s}^n + B_R \hat{\alpha}^n \tag{11}$$

where

$$Z_R \equiv \text{diag}\{z_1, z_2, z_3, \ldots, z_R\} \tag{12}$$

$$B_R \equiv [e^L]^T B_{eq}. \tag{13}$$

Flow values on the boundary can also be obtained directly from the reduced coordinate vector by using the following relationship, obtained by substituting Eq. 10 into Eq. 5:

$$\hat{q}_e = D_R \hat{s} + \varepsilon \hat{\alpha} \tag{14}$$

where

$$D_R \equiv C^{-1} D e^R. \tag{15}$$

If Eqs. 11 and 14 are applied directly, with a small value of R, the calculated system response may be considerably in error. This is because components of the forcing parallel to the omitted eigenmodes are neglected. The use of a quasi-static correction may substantially reduce this error. This technique is similar to the mode-acceleration method common to structural dynamics.[9,10]

With the quasi-static correction, it is assumed that the dynamics of the system can be approximated by the first R eigenmodes. The remaining $N - R$ modes respond in only a pseudo-static manner. Therefore, it can be shown that the time dependent perturbation flow field can be found by combining that due to the first R eigenmodes, $\hat{q}_R$, and that due to the error in the instantaneous forcing $\hat{q}_{err}$. The perturbation flow field due to the forcing error can easily be found from

$$\hat{q}_{err}^n = [I - A_{eq}]^{-1} B_{eq} \hat{\alpha}^n - \sum_{i=1}^{R} e_i^R e_i^{L^T} \frac{B_{eq} \hat{\alpha}^n}{z_i - 1} \tag{16}$$

As the reader can see, the basic idea is beautifully simple. The "devil is in the details," which is why it has taken several years to convert this idea into reality as described in the subsequent sections of this paper and as fully appreciated by those who have labored toward this goal of reduced-order modeling.

Fluid Models

Several fluid models have been considered to date. These range from classical, incompressible, potential flow models to compressible, rotational models (Euler equations). Exploratory work has been completed with viscous, Navier–Stokes models, though much remains to be done there. Most results have been for two-dimensional flows over airfoils and cascades. Only for incompressible, vortex lattice models have three-dimensional flows over wings been considered. No special conceptual difficulty is anticipated in extending other flow models to three dimensions. Clearly the work needs to be done.

We will defer a discussion of the details of the several fluid models to the subsequent section on results. In this section, an important conceptual categorization is discussed based upon dynamical systems ideas that transcend all fluid models, i.e., fully linear models, dynamically linear models, and fully nonlinear models.

Fully Linear Models

Most classical aerodynamic models fall into this category, e.g., small perturbation theory in subsonic or supersonic flow that leads to a form of the convected Laplace or wave equation (with constant coefficients) for, say, the velocity potential. Although such models were not the primary motivation for our work on eigenmodes, it turns out the reduced-order models for direct solution of aeroelastic eigenvalues is a powerful and computationally efficient approach for fully linear models as well.

Dynamically Linear Models

In transonic flow, for example, even potential flow models must be considered in their nonlinear form. Physically this is because the variation of the flowfield over a nonlifting airfoil has a significant effect on the lift on the airfoil when it oscillates. In other words, the nonlifting (static) flowfield is inherently nonlinear and it must be determined from the static solution of the full, nonlinear potential model. Fortunately, however, the lift due to the airfoil oscillations (for sufficiently small motion) may be treated as a linear dynamical perturbation about the nonlinear static or steady flowfield.

This basic idea extends to Euler and Navier–Stokes flows as well. Hence, eigenmode analysis may still be used, but the eigenmodes must be those of a small perturbation with respect to the appropriate nonlinear static flowfield.

As an aside, it might be noted that this latter restriction can likely be relaxed in the following way. If we determine the eigenvalues and eigenmodes about one airfoil at one Mach number, it is likely these might be used, with good accuracy and efficiency, to form a modal series representation for another not too dissimilar airfoil at a not too removed Mach number. Of course, the coefficients of the modal expansions would be different for the two airfoils or the two different Mach numbers. When eigenmodes for one physical system are used to represent the solution of another physical system, these are usually referred to as primitive modes. The use of primitive modes is well established in the aeroelastic iterature, see Refs. 9–13, for example.

When might the use of primitive modes fail? One case might be, if the eigenmodes of a shockless flow were used to represent the flow about an airfoil with shocks. Clearly, if two flowfields are qualitatively different, using the eigenmodes of one to represent the flowfield of the other is problematical as a practical matter.

It should be noted that for an elastic wing, the static wing shape may be changed by the aerodynamic flow, and thus the static wing shape may vary with dynamic pressure. Therefore, the aeroelastician may need to determine the aerodynamic eigenmodes for several dynamic pressures for some applications. In the design of an aircraft or wind tunnel model, however, usually the static shape itself is a design goal and, hence, in that sense, it is known a priori.

Fully Nonlinear Models

In aeroelastic models, the dominant nonlinearity may be either structural or aerodynamic. Clearly, if a structural nonlinearity is dominant, which is not infrequently the case, then a dynamically linear aerodynamic theory is perfectly adequate to determine not only the onset of flutter, but also the limit cycle oscillations that may exist. Note, moreover, that the aerodynamic eigenmode approach is equally suitable for the construction of either time-domain or frequency-domain aerodynamic models. Thus aerodynamic eigenmodes are particularly useful for nonlinear aeroelastic analyses when combined with nonlinear structural models. By contrast, classical aerodynamic models usually provide results in the frequency domain and CFD models normally generate results in the time domain. Of course, in principal and with some effort in practice, the classical aerodynamic and CFD models may be used in either frequency or time domains, though with less ease than the eigenmode approach.

Regarding aerodynamic nonlinearities, these most often may be due to shock waves or separated flow. Note, however, that the presence of a shock wave or separated flow does not per se dictate that the flow is dynamically nonlinear. For sufficiently small airfoil or wing motions, dynamic linearization of the aerodynamic flow is still a valid approximation. Of course, the presence of shock waves or separated flow does mean the steady (i.e., static) flow equilibrium is nonlinear. See, for example, Ref. 16 as an example of a dynamically linear CFD approach that includes the effects of shock waves.

It might be thought that eigenmodes could not be used for fully nonlinear models but, of course, they can be with some additional work. Again, to be concrete, if an airfoil undergoes large motions, a fully nonlinear dynamic flow model must be used to describe the corresponding flowfield.

How then could eigenmodes be useful? Conceptually, the idea is as follows. This procedure will be familiar to those structural dynamicists and aeroelasticians who have studied structural nonlinearities for plates and shells or helicopter blades.[17,18]

First one considers small motions and determines the eigenvalues and, very importantly, the eigenmodes. One then forms a dynamical coordinate transformation from the (generalized coordinate) unknowns of the nonlinear dynamical system to so-called normal mode coordinates, e.g., recall Eq. (10) where q are the original unknowns, $[e^R]$ is a matrix whose columns are the (right) eigenvectors of the eigenmodes and s are the normal mode coordinates. This equation essentially defines s in terms of q. Substituting this equation into the full nonlinear equations of fluid motion for q, recall Eq. (1), premultiplication of the result by $[e^L]^T$ and then truncating to a small, finite number of s will produce a nonlinear, reduced-order model. Unlike the corresponding linear models where the equations for s will be uncoupled, however, now the equations for s will be coupled due to the nonlinear terms. Even so, the number of equations to be solved for a given level of accuracy in determining the flowfield will be much smaller than the original number of equations for the q unknowns. Hence, a very substantial savings in computational cost will still be realized. Indeed, it is for fully nonlinear dynamical models where the full power of the eigenmode approach may be realized. A formal dynamically nonlinear second-order theory has been constructed extending the analysis of Eqs. (1–16).

Two final points are worthy of mention. For nonlinear dynamical systems, it is possible to extend the idea of a linear eigenmode itself to nonlinear eigenmodes. This has some theoretical interest. These nonlinear eigenmodes still lead to coupled equations, however, so their value in practice is often not substantially greater than that of linear eigenmodes. Even so, some day this should and will be investigated.

Finally, it is worth noting that if one determines the linear eigenmodes for, say, one airfoil/Mach number combination and then uses the transformation from q to s for another airfoil/Mach number combination, then the corresponding equations for s will also be coupled even in the linear terms. This, of course, is because we have used the eigenmodes of one fluid system to represent the dynamics of a different fluid system. That is, orthogonality of the modes only holds for eigenmodes used for the same dynamical system from which they were derived.

Eigenmode Computational Methodology

For the simpler (lower dimensional) fluid models, say, the vortex lattice model, the size the eigenvalue matrix is of the order 100×100. For such matrices, standard eigenvalue extraction numerical procedures may be used. We have used EISPACK, an algorithm and computer code available in most computational centers in the United States.

For more complicated fluid models, e.g., the full potential models or Euler models, the order of the eigenvalue matrix may be in the range of $1,000 \rightarrow 10,000$ or greater.[2] For matrices of this size, new developments in eigenvalue extraction have been required. We have used methods based on the Lanczos algorithm. For the full potential equation ($\sim 1000 \times 1000$) an efficient and effective algorithm is described by Hall et al.[3] For the Euler equations, a forthcoming paper by Dowell and Romanowski[6] will be of interest. The discussion of Mahajan et al.[4]

is also recommended to the reader. As the extensions to three-dimensional and viscous flows are made, further developments in eigenvalue and eigenmode determination will likely be required or desired.

These further developments appear doable, but the amount of work should not be underestimated. Perhaps an appropriate use of primitive modes may be of help. That is, it may be possible to use the eigenmodes from a simpler fluid model as primitive modes for a more advanced fluid model. Much work remains to be done here.

A final and important point that Ref. 4 has emphasized is that the eigenvalue problem may be formulated in either discrete or continuous time. The former allows eigenvalue extraction from existing CFD codes using pre- and postprocessor formats; thereby saving the considerable effort of recoding existing CFD codes.

Representative Results and Key Insights

Classical Incompressible, Potential Flows

The discussion begins with results from simpler fluid models and proceeds to the more advanced models. In his seminal paper, Hall[1] has used a vortex lattice model to describe the eigenmodes of inviscid, incompressible, irrotational, two-dimensional and three-dimensional flows about airfoils and wings. Hall displays the eigenvalues in terms of both z and λ where

$$z = e^{\lambda(\Delta t U/c)}$$

with c the airfoil chord. To quote Hall,

> ... the eigenvalues are lightly damped, and form a dense line which runs close to the imaginary axis in the λ-plane. The line of eigenvalues intersects the real axis very near the origin, and may be thought of as approximating a branch cut of the aerodynamic transfer function. Numerical experiments reveal that this line of eigenvalues gets denser as the length of the computational wake is increased with constant element size Δx. The line of eigenvalues gets longer as the element size Δx is reduced with constant wake length.
>
> The presence of a branch cut is consistent with the well-known result that the Theodorsen function contains a branch point at the origin.

The corresponding results for a rectangular wing of aspect ratio 5.0 were also determined by Hall.

Hall then used the eigenvalues and eigenmodes so determined to construct a reduced-order model and compared the results obtained with the reduced-order model to those obtained using conventional methods for an oscillating airfoil and wing over a substantial range of reduced frequency. The airfoil and wing had total degrees of freedom of 220 and 480, respectively. The point to be emphasized here is that using only 40 or fewer eigenmodes ($m \leq 40$), the reduced-order model gave essentially the same accuracy as the original model for a reduced frequency up to 1.5.

Indeed, if results were only needed for a smaller range of reduced frequency, say, $k < 0.5$, then fewer than 10 modes are sufficient. Note, in particular, that the number of eigenmodes needed to obtain a given level of accuracy for the lift is no greater for the wing than for the airfoil. This is most encouraging.

Finally, for the airfoil free to move in plunge and pitch, Hall performed a classical bending/torsion typical section flutter analysis using the reduced-order aerodynamic model and calculating the true aeroelastic eigenvalues (called a p method, by aeroelasticians) and compared the results to a conventional $V-g$ method. The latter method, of course, gives only a true, meaningful physical result when the damping is zero in the flutter mode at the flutter (neutrally stable) condition. Results were obtained for the complete fluid-structural model ($m = 220$), as well as from a reduced-order model ($m = 40$). The $m = 220$ and 40 results are in near perfect agreement and also agree with the $V-g$ result at the flutter condition. Note, however, that significantly different results were obtained away from the flutter condition. The eigenmode results are physically correct at all velocities, whereas the $V-g$ method is only correct at the flutter point per se.

Compressible Potential Flow with a Nontrivial (Nonlinear) Steady Flow

Florea and Hall[2] have presented results for this fluid model using a finite element method based on a variational principle. Mahajan et al.[4] have also considered this type of flow model. There are some distinct facets of their work that are worthy of special mention here. First of all, there are some technical differences, e.g., in Ref. 4 a finite volume numerical model is used for the full potential equation and a somewhat different algorithm is used to extract the eigenvalues.

The point to be emphasized here, however, is that in Ref. 4 the numerical fluid model has been combined with a typical section, bending torsion structural model (for a cascade) and the eigenvalues for the full fluid-structural (aeroelastic system) have been successfully extracted. Again the aeroelastic or flutter results of this new (p) method have been compared to those from a conventional ($V-g$) method and the conclusions drawn are the same as those found for classical, potential flow.

Compressible, Euler Flow with a Nontrivial (Nonlinear) Steady Flow

Mahajan et al.[4] pioneered eigenmode studies for this flow model, see Ref. 4 and other references therein to earlier work by these authors. Building on this base, Romanowski and Dowell[5] have reported the first successful construction of reduced-order aerodynamic models for Euler flows.

As a reference, Fig. 1 shows the results for lift on an airfoil due to step change in angle of attack using conventional CFD solution methods. Results are shown both for a dynamical linear [linearized Sankar-Tang (ST) code] model and a full nonlinear model (ST code). See references to Sankar's work.[4,5] Note that for the conditions shown the two results agree very well.

Next, the reduced-order model was constructed and the results compared to those in Fig. 1. In Fig. 2 results are shown for all (1184), 200,100, and 50 modes. At large times, all of these results agree, say, for $\tau \equiv (U_\infty t/c)/M > 4$ where $M = 0.5$ is the Mach number. For smaller time where the peak in lift occurs near $\tau = 0.5$, the larger the number of modes, the better this peak is replicated. The inclusion of 200 modes does very well. In Fig. 3, a comparison of the exact results and a single-mode reduced-order model is shown to make the point that for large time only a few modes are needed to give good accuracy.

Recalling the Fourier transform relationship between large time and small frequency, it might be anticipated that for sufficiently small reduced frequency, a small number of eigenmodes would suffice. This is confirmed by the results of Figs. 4 and 5 (Figs. 4 and 7 of Ref. 5). At a reduced frequency of $k = 0.4$ only one mode gives quite reasonable results for this airfoil at this Mach number. Of course, other airfoils or Mach numbers may require larger numbers of eigenmodes and that will certainly be the case at higher frequencies.
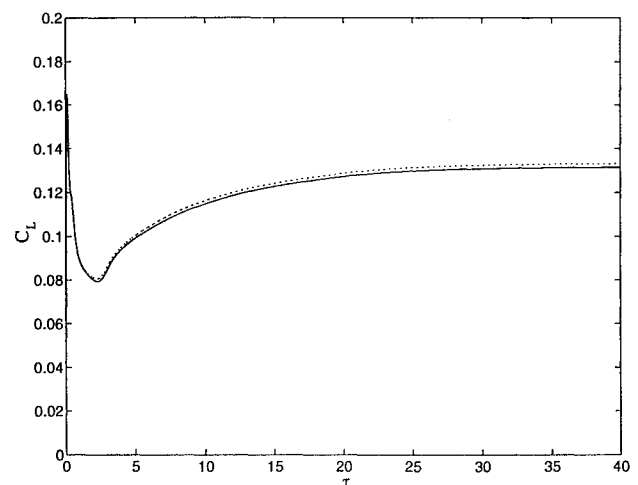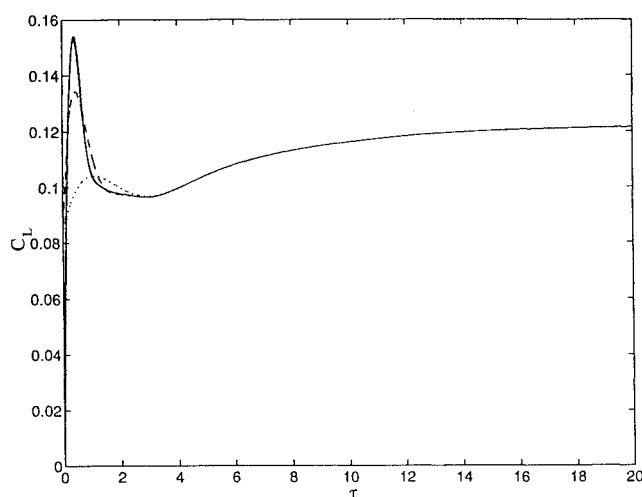

Fig. 1   Lift response vs time for a NACA 0012 airfoil at Mach 0.5 and zero angle of attack, subjected to a 1-deg step change in angle of attack: ST code (-), linearized ST code ($\cdots$), $\varepsilon_e = 2$, $\varepsilon_i = 0$ and $\Delta\tau = 0.004$.

Fig. 2 Lift response vs time for a NACA 0012 airfoil at Mach 0.5 and zero angle of attack, subjected to a 1-deg step change in angle of attack; coarse mesh, $\varepsilon_e = 2$, $\varepsilon_i = 0$, $\Delta\tau = 0.004$, linearized ST, all modes, and 200 modes overlay (—), 100 modes (—), 50 modes (· · ·).
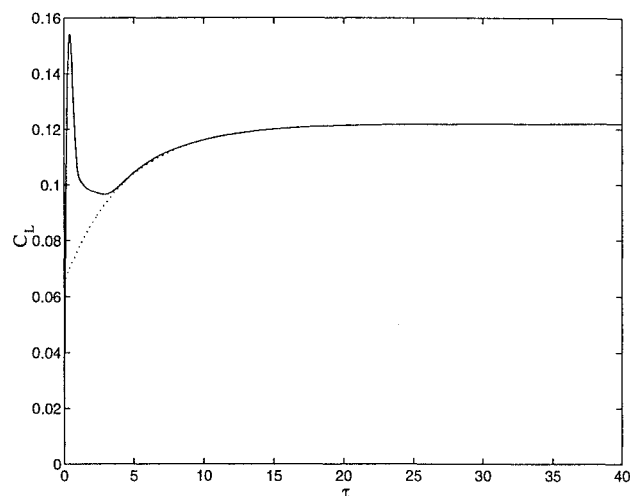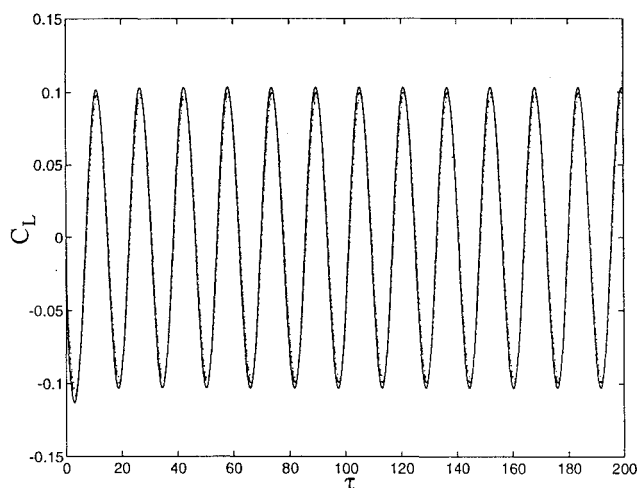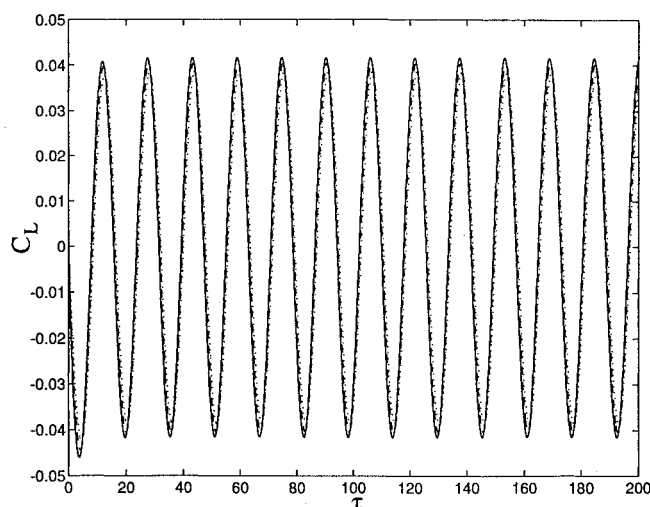

Fig. 3 Lift response vs time for a NACA 0012 airfoil at Mach 0.5 and zero angle of attack, subjected to a 1-deg step change in angle of attack; coarse mesh, $\varepsilon_e = 2$, $\varepsilon_i = 0$, $\Delta\tau = 0.004$, linearized ST (—), 1 mode (· · ·).

Fig. 4 Lift response vs time for a NACA 0012 airfoil at Mach 0.5 oscillating at ± 1 deg about zero angle of attack; $k = 0.4$, coarse mesh, $\varepsilon_e = 2$, $\varepsilon_i = 0$, $\Delta\tau = 0.004$; linearized ST, and 100 modes overlay (—), 50 modes (—), 10 modes (· · ·).

Fig. 5 Lift response vs time for a NACA 0012 airfoil at Mach 0.5 in plunging oscillation; $h/c = 0.01$, $k = 0.4$, coarse mesh, $\varepsilon_e = 2$, $\varepsilon_i = 0$, $\Delta\tau = 0.004$; linearized ST and 50 modes overlay (—), 10 modes (—), 1 modes (· · ·).

Concluding Remarks Including Comments on Future Directions

So where do we stand and where might we go?

Where we stand is that a powerful new approach to modeling unsteady aerodynamic flows has been developed. It will provide a level of accuracy and computational efficiency not previously available. In particular, for construction of reduced-order models based upon rigorous fluid dynamical theory is now possible to 1) calculate true damping and frequency for all aeroelastic modes at all parameter conditions, 2) provide a practical approach for constructing highly efficient, accurate aerodynamic models suitable for designing control laws and hardware for aeroelastic systems, and 3) make the use of CFD models routine in aeroelastic analysis.

This is a considerable achievement. What might the future bring? For fully (dynamically) nonlinear models, we should be able to develop rigorous reduced-order models that will accurately model large and violent aircraft motions. For aeroacoustics, the eigenmode/reduced-order model should work well here also, but far-field boundary conditions will need special attention for this (or any other) approach. See Ref. 19 for a discussion of the present state of the art in computational aeroacoustics. For turbulence and turbulence models, if we use a standard turbulence model, e.g., k–ε, etc., then the present method formally goes through. However, it is possible that the real value of the eigenmodel reduced-order model approach will be to encourage the development of better turbulence models.

Is it possible that one could attack the full Navier–Stokes equations using the eigenmode/reduced-order model methodology? The answer is that in some sense such work has already begun. The classical hydrodynamical stability theory is based on the boundary-layer approximation combined with a highly simplified geometry, a flat plate of infinite extent. However, that work per se, now some 50–70 years ago in its origins, did not lead to advances beyond the limitations of the classical infinite geometry. Using an outer inviscid model combined with viscous boundary-layer theory, one might cautiously hope to overcome that classical geometrical limitation and treat the larger scale viscous motions about an airfoil or wing. With these large-scale motions determined, it might even be possible to refine the eigenmode representation for local flow behavior. Clearly this is only a hypothesis, but a very intriguing one.

Finally, for those (including the author) who still find fascination and challenge in the classical models, it would be very interesting to explore the question of in what sense do discrete, but closely spaced, eigenvalues represent a branch cut in two- and three-dimensional, fully linearized potential flows?

In 5 or 10 years, some of these questions should have definitive answers and very likely others will supersede them. If the reader is

encouraged or inspired to explore eigenmodes for unsteady aerody-
namic flows, this paper will have served its purpose.

Acknowledgments

References

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